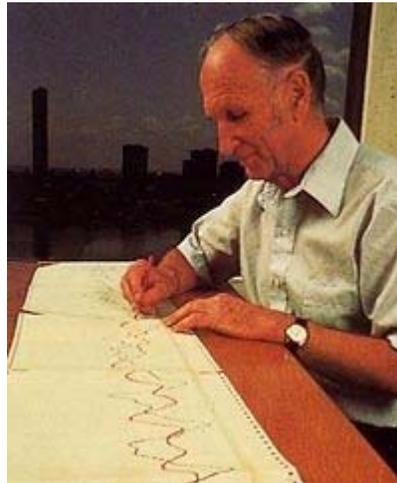


DYNAMICS OF THE LORENZ EQUATIONS

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0a Content

Chapter 1: Introduction to the Lorenz Equations	1
Introduction	1
Historical Setting	1
Derivation of Lorenz Equations	2
Properties of the Lorenz Equations	3
Dissipative	4
Symmetry	5
Chapter 2: Local Bifurcation Theory	6
Preliminaries	6
Bifurcation	6
Types of Bifurcations	7
Pitchfork Bifurcations	8
Hopf Bifurcation	9
Chapter 3: Steady States of the Lorenz Equations	11
Steady State Solutions	11
Stability of the Point C_0	12
The Stability of the Points C_1 and C_2	14
Summary	15
Chapter 4: The Numerical Approach	17
Introduction	17
Approaching Chaos	17
Shifting the Chaotic Boundary	20
Sensitivity to Initial Conditions	21
Chaos Reigns	22
Chapter 5: Further Interesting Properties	23
Preliminaries	23
Period Doubling Windows	23
Intermittent Chaos	26
Homoclinic Explosions and Small Values of b	27
Chapter 6: Conclusion	29
Conclusion	29
Afterword	29
Chapter 7: References	30
Chapter 8: Appendix A	31

0b Figure Content

Chapter 1: Introduction to the Lorenz Equations	1
Figure 1.1	1
Figure 1.2	5
Chapter 2: Local Bifurcation Theory	6
Figure 2.1	7
Figure 2.2	8
Figure 2.3	9
Chapter 3: Steady States of the Lorenz Equations	11
Figure 3.1	13
Figure 3.2	16
Chapter 4: The Numerical Approach	17
Figure 4.1	18
Figure 4.2	18
Figure 4.3	19
Figure 4.4	20
Figure 4.5	21
Figure 4.6	22
Chapter 5: Further Interesting Properties	23
Figure 5.1	24
Figure 5.2	24
Figure 5.3	25
Figure 5.4	25
Figure 5.5	26
Figure 5.6	27
Figure 5.7	28
Chapter 6: Conclusion	29
Chapter 7: References	30
Chapter 8: Appendix A	31

0c Abstract

This paper will explore the Lorenz Equations, thought up by Edward Lorenz in 1963, and described in a paper which changed the mathematical world forever. The author will highlight some important and interesting properties and give rigorous proofs for some of the more obvious ones. The paper is designed to give an introduction to the wonderful world of Chaos. The paper will also introduce the reader to Bifurcations and explain simply what they are and why they exist. Finally the author will use Numerical Methods to find out more about the Lorenz equations.

1 Introduction to the Lorenz Equations

(1.0) Introduction

This paper is designed to discuss some of the most fundamental and interesting properties of the Lorenz equations (to discuss all the properties of the Lorenz equations is far beyond the scope of a single paper). It will be assumed that the reader has no prior knowledge of dynamical systems or bifurcations. It will be assumed, however, that the reader is well versed in algebraic manipulations and derivatives (ordinary and partial).

At this point the author would like to note that all of the computations of the Lorenz equations must be done numerically, as analytical solutions are impossible, using known methods. The author uses MatLab for all the numerical computations and diagrams; however the reader need not be familiar with MatLab as the paper is interested in the properties not the programming.

(1.1) Historical Setting

In 1881 the French mathematician Henri Poincaré published *Mémoire sur les courbes définies par une equation différentielle*, in which he studied the problem of the motion of three objects in mutual gravitational attraction. He found that there can be orbits that are nonperiodic, but not increasing to infinity nor tending to a fixed point. This was the first paper to suggest such an idea. After this point only a few special minds identified more systems showing the same characteristics of nonperiodicity and sensitivity to initial conditions; the overwhelming consensus was that, outside the quantum world, classical physics provided *the* theory for completely predicting the state of the universe at any future time (indeed experimentalists would routinely hide (i.e. not publish) data from real systems that did not conform to this behaviour).

In the middle of the 1900s, computer and satellite technology was being developed and it was believed this would allow the human race to completely predict and control the weather. Clearly, this has not happened because everyone knows that weather forecasting is still not very accurate. The problem was the assumption that tiny perturbations in the system only amount to tiny changes over time.

In 1963 Lorenz published his paper, *Deterministic Nonperiodic Flow*, in which he showed that tiny differences in the initial conditions actually amount to dramatic differences in the systems behaviour over time. As Gleick (Gleick 1987: 21) puts it, “if one infinitely accurate sensor were placed within every cubic foot of the Earth’s atmosphere, and the data were fed to an infinitely powerful computer, reasonable prediction (e.g. rain vs. shine) would still be limited to less than one month”. This is just illustrating that predictions become suddenly truncated even in a completely deterministic system. Still today, modern mathematicians and physicists alike will neglect small nonlinear terms in order to simplify the system. There is a reluctance to abandon the predictability of the classical universe. (This is why mathematicians spend so much time on domains of definition for these approximations).

(1.2) Derivation of Lorenz Equations

In this section the author wishes to give a brief description of where the Lorenz equations come from. One can spend many hours trying to derive them formally, however, the derivation is long winded, complicated and above the scope of this paper. This paper is more interested in the behaviour of the system. (If the reader requires more details regarding the derivation then see Kundu 2002, Lorenz 1963 or Sparrow 1982).

To derive the Lorenz equations one must look to Saltzman paper (Saltzman 1962), as Lorenz did. The convection equations of Saltzman came from the investigation of a fluid of uniform depth H , with a temperature difference between upper and lower layer of ΔT , in particular with linear temperature variation. In the special case where there is no variation with respects to the y -axis, Saltzman provided the governing equations:

$$\frac{\partial}{\partial t} \nabla^2 \psi = - \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, z)} + \nu \nabla^4 \psi + g \alpha \frac{\partial \theta}{\partial x},$$

$$\frac{\partial}{\partial t} \theta = - \frac{\partial(\psi, \theta)}{\partial(x, z)} + \frac{\Delta T}{H} \frac{\partial \psi}{\partial x} + \kappa \nabla^2 \theta,$$

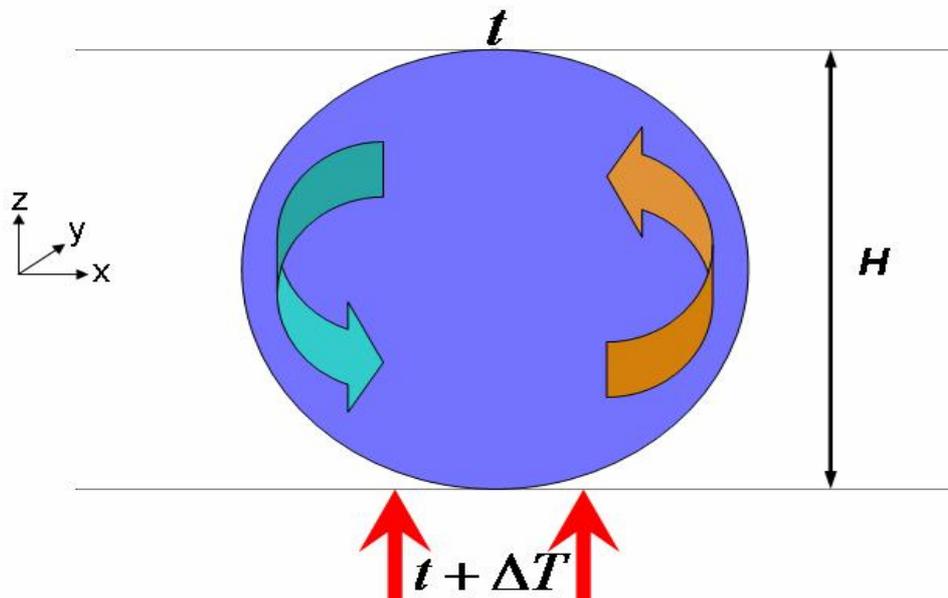


Figure 1.1: The simplified fluid motion described by Saltzman.

where ψ is a stream function for the two-dimensional motion, θ is the temperature deviation from the steady state. ψ , $\nabla^2 \psi$ vanish at the upper and lower boundaries, and g , α , ν , κ are, respectively, constants of gravitational acceleration, coefficient of thermal expansion, kinematic viscosity and thermal conductivity. Rayleigh discovered that at a critical point (known now as the Rayleigh number) these

equations show a convective motion. Lorenz then defined three time dependent variables:

X proportional to the intensity of the convection motion,

Y proportional to the temperature difference between the ascending and descending currents.

Z proportional to the difference of the vertical temperature profile from linearity.

The result, from substituting the above into the Saltzman equations, and further derivation is the Lorenz equations:

$$\frac{dX}{dt} = \sigma(Y - X), \quad (1.1)$$

$$\frac{dY}{dt} = X(r - Z) - Y, \quad (1.2)$$

$$\frac{dZ}{dt} = XY - bZ. \quad (1.3)$$

Where $\sigma = \frac{\nu}{\kappa}$ is the prandtl number (ratio between momentum diffusion, ν , and heat diffusion, κ), $r = \frac{R_a}{R_c}$ is the Rayleigh number over the critical Rayleigh number (a measure of heat into the system) and $b = \frac{4}{(1+a^2)}$ gives the size of the region to be approximated (a comes from the solution for ψ and θ). All the parameters are taken to be positive.

Note that in this paper the author will examine many different parameter values, to investigate some interesting features, but the system is only a realistic model of the intended fluid convection is r is close to 1.

(1.3) Properties of the Lorenz equations

There are far too many properties of the Lorenz equations to place them all in this paper. The author will only highlight the behaviour of a few properties, namely the ones needed later in the paper for further analysis of the Lorenz equations. In this section it will be proved that the Lorenz equations do not tend to infinity, and will show that the system has symmetry. For further information on other properties one could look into Sparrow 1982.

(1.3.1) Dissipative

A system is dissipative if every orbit eventually moves away from infinity. Or more rigorously $\exists B \subset \mathbb{R}^3$ bounded, such that $\forall x_0 \in \mathbb{R}^3 \exists t_0(x_0, B)$ with solution $\varphi(t, x_0)$ satisfying $\varphi(t, x_0) \in B \forall t \geq t_0$. The Lorenz equations can be shown to be dissipative by using one of the Liapunov functions,

$$V = rX^2 + \sigma Y^2 + \sigma(Z - 2r)^2 \quad (1.4)$$

A Liapunov function is a function that allows us to see whether a system has a stable or unstable critical point at the origin, if we have an autonomous system with first order differential equations (as with Lorenz system),

$$\frac{dx}{dt} = F(x, y, z), \quad \frac{dy}{dt} = G(x, y, z), \quad \frac{dz}{dt} = H(x, y, z).$$

A function $V(x, y, z)$ that is one time differentiable (in all variables) and satisfies $V(0, 0, 0) = 0$ is called a Liapunov function if every open ball $B_\delta(0, 0, 0)$ contains at least one point where $V > 0$. The Liapunov function, in this case, is a metric and that is why it has been chosen so that it can show that the Lorenz equations are dissipative, more easily than some of the other Liapunov functions. (More details on why this Liapunov function is the most convenient can be found in Sparrow 1982: 196)

Then,

$$\begin{aligned} \frac{dV}{dt} &= 2rX \frac{dX}{dt} + 2\sigma Y \frac{dY}{dt} + 2\sigma(Z - 2r) \frac{dZ}{dt} \\ &= -2\sigma(rX^2 + Y^2 + bZ^2 - 2brZ). \end{aligned}$$

Choose the bounded region D such that $\underline{X} \in D \Leftrightarrow \frac{dV(\underline{X})}{dt} \geq 0$, and let c be the maximum of V in D . Let E be the sphere defined by $V \leq c + \varepsilon$ for small $\varepsilon \geq 0$. Then $\underline{X} \notin E \Rightarrow \underline{X} \notin D \Rightarrow \frac{dV(\underline{X})}{dt} \leq -\delta$ for some $\delta > 0$, and the points on the trajectories passing through \underline{X} will be associated with a decreasing V . Thus the trajectories will eventually enter and remain in E . The author does not have a rigorous proof, at this time, that this is a unique solution, however in the numerical computations done later in the paper no evidence has been found to the contrary.

It follows from the fact that the divergence of the system is negative, $-(1+b+\sigma)$, that the volume of this region will decrease at a rate of $\exp[-(1+b+\sigma)t]$, so the set towards which all trajectories tend has zero volume.

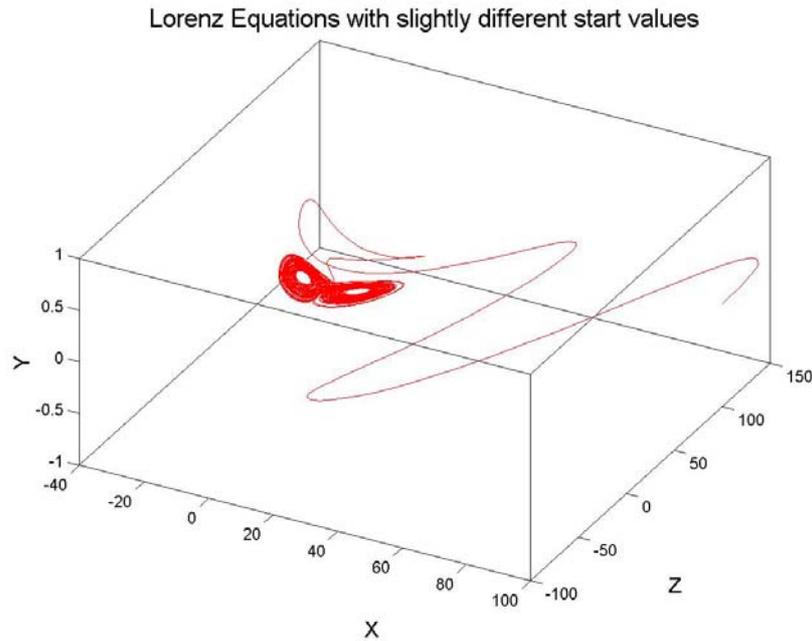


Figure 1.2: A graph of the Lorenz system, numerically computed using MatLab, starting at $(100,0,100)$ with values of constants being $\sigma = 10, b = \frac{8}{3}, r = 28$.

(1.3.2) Symmetry

The Lorenz equations are invariant under the following transformation:

$$(X, Y, Z) \mapsto (-X, -Y, Z). \quad (1.5)$$

This can be seen by simple substitution of the transformation into the Lorenz equations,

$$-\frac{dX}{dt} = \sigma(-(-X) + (-Y)) \Rightarrow \frac{dX}{dt} = \sigma(Y - X)$$

$$-\frac{dY}{dt} = r(-X) - (-Y) - (-X)Z \Rightarrow \frac{dY}{dt} = X(r - Z) - Y$$

$$\frac{dZ}{dt} = -b(Z) + (-X)(-Y) \Rightarrow \frac{dZ}{dt} = XY - bZ$$

The invariance of the Z-axis implies that all trajectories on the Z-axis remain on the Z-axis, and approach the origin (See Chapter 3). Furthermore, since

$$X = 0, Y > 0 \Rightarrow \frac{dX}{dt} > 0 \text{ and } X = 0, Y < 0 \Rightarrow \frac{dX}{dt} < 0$$

All trajectories that rotate about the Z-axis must move clockwise with increasing time (looking from above onto the XY plane).

2 Local Bifurcation Theory

(2.0) Preliminaries

This chapter is designed to give the reader an insight into the beautiful world of local bifurcation theory. It is by no means the full story on the theory it is just the basic properties. Though, it should be enough to enable the reader to understand what is going on with the Lorenz equations. (If more information on the theory is needed then one could look into Gleick 1987, Hale and Kocak 1991, Guckenheimer and Holmes 1990).

(2.1) Bifurcation

In simplest terms a bifurcation is a separation of a structure into two branches or parts. If one was to watch a tap dripping, one may have noticed the ‘double drip’ phenomena, this is where two drips fall in quick succession and then a longer pause then another two drips. If the water pressure was to be increased then one would observe the ‘quadruple drip’ phenomena, and so on until the system reaches chaos (when the bifurcations become so close together the observer cannot distinguish between them).

More formally the definition of a bifurcation can be given as:

“A bifurcation can be a qualitative change of the attractor of a dynamical system as the result of a moving parameter”.

The above definition is a subtle one, but will become clear by the end of the chapter. One analogy one could use to visualise chaos is the probability tree for tossing a coin (it is not chaos or any relation to it, however illustrates the sort of form or diagram one might expect with chaos). The tree begins with just two branches and after each toss the branches will divide into two, hence how four branches and so on. Of course this is just to work out the probabilities of obtaining, say three heads in a row, but the diagram seems to get very ‘chaotic’ after a large number of tosses, as figure 2.1 should show.

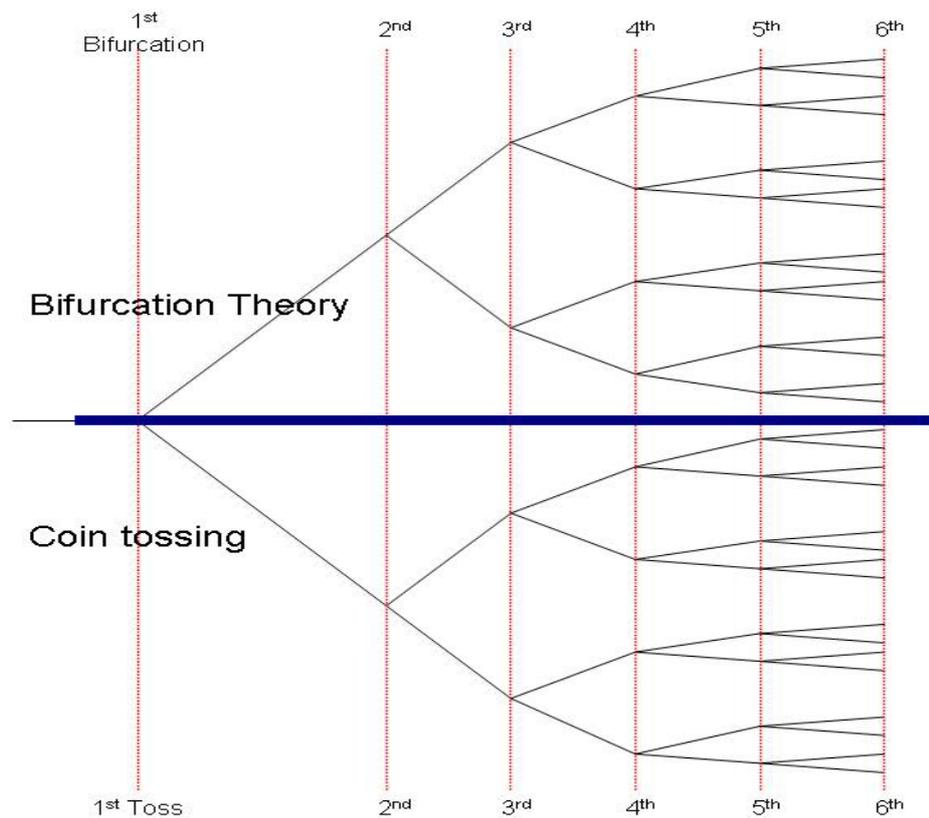


Figure 2.1: A coin tossing probability tree analogy to bifurcation theory.

(2.2) Types of Bifurcations

Now that it has been established that bifurcations exist, the next sensible thing to think about is; how many different types of bifurcations exist? To answer this question the author will just state that there are four different types. One could do the analysis to prove this to be true, but again it is far beyond the level of this paper, and besides it does not help the study of the Lorenz equations.

The four types of bifurcations are:

1. Pitchfork Bifurcation
2. Hopf Bifurcation
3. Transcritical Bifurcation
4. Limit Point (Saddle Node) Bifurcation

Transcritical and Limit point bifurcations will not be discussed in this paper, but if the reader wants more information they can look into Gleick 1987, Hale and Kocak 1991, Guckenheimer and Holmes 1990.

Note that there two different versions of each of the above bifurcations, namely subcritical and supercritical bifurcations. Subcritical means that the bifurcation happens at a critical point (and creates numerous trajectories) but is stable away from that point (trajectories). Whereas, supercritical means that the bifurcation occurs at a critical point (and creates numerous trajectories) and is unstable about that critical

point (trajectories). The subtle differences can be easily shown by a ‘sketch’ graph of a typical bifurcation, and its sub / supercritical versions, as in figure 2.2.

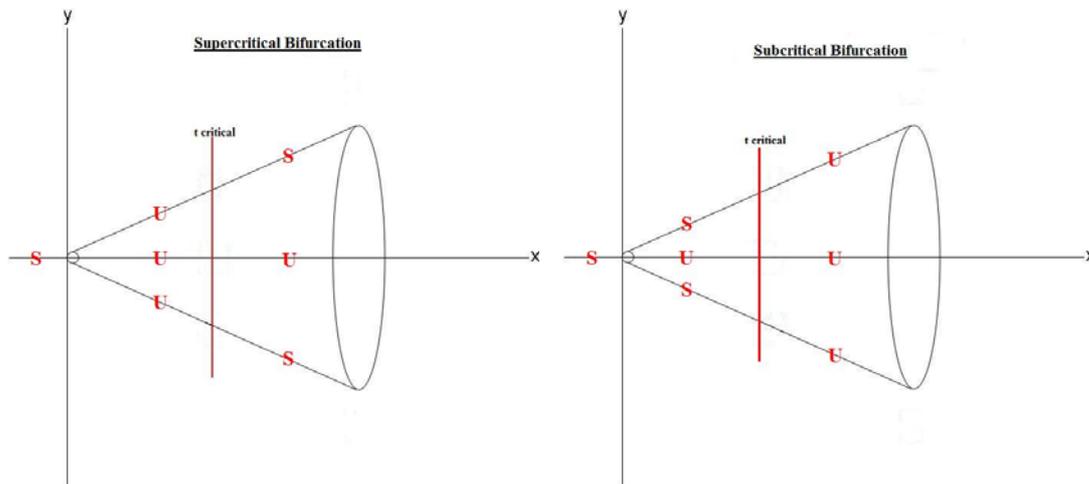


Figure 2.2: Diagrams illustrating the difference between Subcritical and Supercritical bifurcations.

(2.3) Pitchfork Bifurcation

In this section the most fundamental properties will be discussed. The author will try to explain some of the properties with diagrams as well. The Pitchfork bifurcation gets its name simply because it looks like a Pitchfork. The standard form of the Pitchfork bifurcation is:

$$\frac{dx}{dt} = x(a - bx^2) \quad (2.1)$$

From the standard form it is obvious that the steady state solutions are at $x = 0, \pm\sqrt{\frac{a}{b}}$, and so we know there are three solutions.

Notice that there is always three solutions, but they are not all stable at the same time, but it will not be discussed presently because the whole purpose of chapter 3 is to enlighten the reader to the stability of these points with regards to the Lorenz equations.

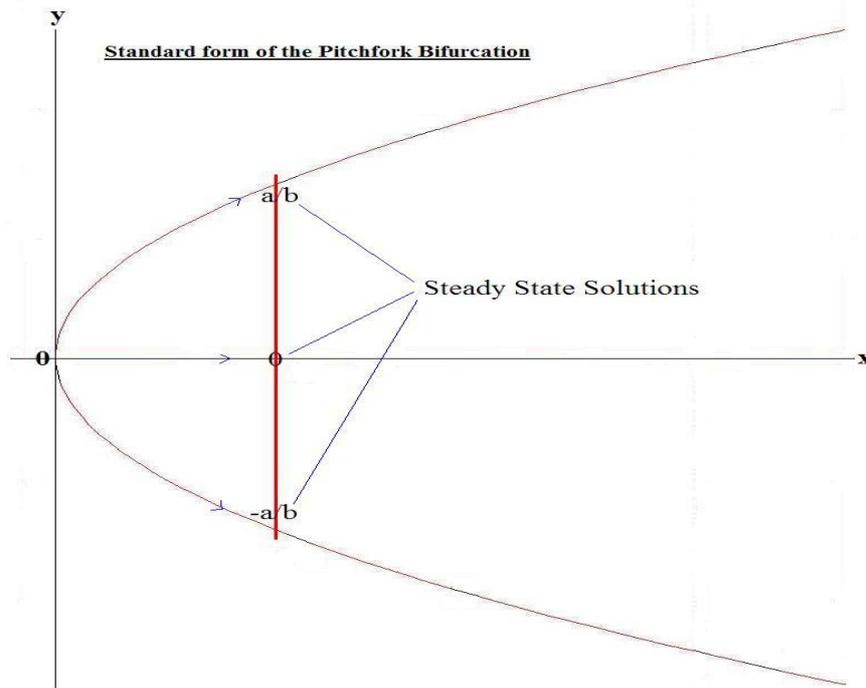


Figure 2.3: A sketch graph showing the standard form and steady state solutions of the Pitchfork Bifurcation.

(2.4) Hopf Bifurcation

The Hopf bifurcation can be described as the bifurcation of a fixed point to a limit cycle (Hale and Kocak 1991). It, like the Pitchfork, has a standard form, but this time it is usually given by two differential equations:

$$\frac{dx}{dt} = -y + (a - x^2 - y^2)x \quad (2.2)$$

$$\frac{dy}{dt} = x + (a - x^2 - y^2)y \quad (2.3)$$

One can, actually, convert this pair of differential equations by considering the complex number $z = x + iy$, and thus $\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$, and so we obtain the differential equation,

$$\begin{aligned} \frac{dz}{dt} &= i(x + iy) + (a - (x^2 + y^2))(x + iy), \\ \frac{dz}{dt} &= iz + (a - |z|^2)z. \end{aligned} \quad (2.4)$$

Now that the single differential equation for the standard form has been discovered, one can extend what is known about this bifurcation, by letting $z = e^{i\theta}$ (and $\frac{dx_i}{dt} = \dot{x}_i$), to give;

$$\begin{aligned} \frac{dz}{dt} &= e^{i\theta} (\dot{r} + ir\dot{\theta}) \text{ and (2.4)} \Rightarrow \frac{dz}{dt} = e^{i\theta} [ir + (a - r^2)r] \\ &\Rightarrow \dot{r} + ir\dot{\theta} = ir + (a - r^2)r. \end{aligned}$$

Thus, comparing real and imaginary parts one will obtain:

$$\dot{r}(t) = [a - r^2(t)]r(t) \quad (2.5)$$

$$\dot{\theta}(t) = 1. \quad (2.6)$$

If (2.5) is compared to (2.1) one should see that they are equivalent, i.e. (2.5) describes a Pitchfork bifurcation. (2.6) describes the Hopf bifurcation rotating at a constant rate. One can solve (2.5) analytically, from noting that;

$$\frac{1}{r^3} \frac{dr}{dt} = \frac{a}{r^2} - 1 \Rightarrow -\frac{1}{2} \frac{d}{dt} \left(\frac{1}{r^2} \right) = a \left(\frac{1}{r^2} \right) - 1,$$

and then solving for the variable $\left(\frac{1}{r^2} \right)$, as follows: Let $u = \frac{1}{r^2} \Rightarrow \frac{du}{dt} + 2au = 2$, and

this is simply solved using an integrating factor $e^{\int 2adt} = e^{2at}$. All that is required to do then is to substitute into the ODE, rearrange and transform back into terms of r , like so:

$$\frac{d}{dt} (ue^{2at}) = 2e^{2at} \Rightarrow u = \frac{e^{-2at}}{a} (e^{2at} + aK) \Rightarrow r^2 = \frac{ae^{2at}}{(e^{2at} + aK)}.$$

So it is known where r is for all time, and hence the solutions should be able to be plotted relatively easily.

3 Steady States of the Lorenz Equations

(3.0) Steady state solutions

Using the techniques derived for fluid mechanics, and considering the variables to be vectors, we set $\frac{\partial}{\partial t} \equiv 0 \Rightarrow \dot{X} = \dot{Y} = \dot{Z} = 0$ (steady state), and let $\underline{x} = (X, Y, Z)$, with the steady solutions at the position vector $\underline{x}_0 = (X_0, Y_0, Z_0)$. Using the above information one can see that,

$$(1.1) \Rightarrow X = Y, (1.3) \Rightarrow \frac{X^2}{b} = Z, (1.2) \Rightarrow X[X^2 - b(r-1)] = 0,$$

with this one deduces that there must be three steady state solutions, at the position vectors

$$\underline{x}_0 = (0, 0, 0) \text{ and } \underline{x}_0 = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1).$$

It seems a sensible progression at this stage to investigate the stability of these steady state solutions, and the effect of changing the variables σ , r and b . The author will call the three steady state solutions, for convenience; C_0 , the trivial solution at the origin, C_1 and C_2 , the positive and negative other two solutions respectively.

Let $\underline{x}_0 + \underline{\varepsilon}(x, y, z) = (X_0 + \varepsilon(x), Y_0 + \varepsilon(y), Z_0 + \varepsilon(z))$ so that quadratic terms of ε can be ignored. Hence the following linearized equations are obtained, (1.1) $\Rightarrow \frac{dx}{dt} = \sigma(Y_0 - X_0) + \sigma(\varepsilon(y) - \varepsilon(x))$ but it is known that $Y_0 - X_0 = 0$, from above, so the equation reduces to

$$\frac{dx}{dt} = \sigma(\varepsilon(y) - \varepsilon(x)). \quad (3.1)$$

Also, (1.2) $\Rightarrow \frac{dy}{dt} = [X_0(r - Z_0) - Y_0] + \varepsilon(x)[r - Z_0 - \varepsilon(z)] - \varepsilon(z)X_0 - \varepsilon(y)$ and here there is $X_0(r - Z_0) - Y_0 = 0$ and $\varepsilon(x).\varepsilon(z)$ is neglected so the equation reduces to

$$\frac{dy}{dt} = \varepsilon(x)[r - Z_0] - \varepsilon(z)X_0 - \varepsilon(y). \quad (3.2)$$

From (1.3) $\Rightarrow \frac{dz}{dt} = [X_0Y_0 - bZ_0] + \varepsilon(y)[X_0 + \varepsilon(x)] + \varepsilon(x)Y_0 - \varepsilon(z)b$, where $X_0Y_0 - bZ_0 = 0$ and $\varepsilon(x).\varepsilon(y)$ is neglected so the solution reduces to

$$\frac{dz}{dt} = \varepsilon(y)X_0 + \varepsilon(x)Y_0 - \varepsilon(z)b. \quad (3.3)$$

(3.1) The Stability of the point C_0

Now that the general forms of the linearized equations have been found, one can study the three steady solutions. First the author studies the steady solution point, C_0 , $\underline{x} = (0,0,0)$ to see what behaviour it exhibits. On substitution the three linearized equations become:

$$\frac{dx}{dt} = \sigma(\varepsilon(y) - \varepsilon(x)) \quad (3.4)$$

$$\frac{dy}{dt} = \varepsilon(x)r - \varepsilon(y) \quad (3.5)$$

$$\frac{dz}{dt} = -\varepsilon(z)b \quad (3.6)$$

$$(3.6) \Rightarrow \int_0^z \frac{1}{\varepsilon(z)} dz = \int_0^t -b dt \Rightarrow \ln[\varepsilon(z)] = -bt + c \Rightarrow \varepsilon(z) = \exp[c - bt] = A \exp[-bt],$$

so it can be shown that

$$\varepsilon(z) = A e^{-bt}. \quad (3.7)$$

For $t \geq 0 \Rightarrow z \rightarrow 0$ as $t \rightarrow \infty$, implying the Z-direction is stable under small perturbations about the point C_0 . (3.4) and (3.5) are ordinary differential equations (ODEs) of two variables, the easiest way to solve them is to assume that their solutions are of the same form as (3.7), namely $\varepsilon(x) = B e^{\lambda_1 t}$ and $\varepsilon(y) = C e^{\lambda_2 t}$, it will be further assumed that $\lambda_1, \lambda_2 \neq 0$ and $B, C \neq 0$. So on substitution of the above assumptions into (3.4) we get

$$B(\sigma + \lambda_1) e^{\lambda_1 t} = C \sigma e^{\lambda_2 t},$$

which is only consistent if $\lambda_1 = \lambda_2 = \lambda$, because if $\lambda_1 \neq \lambda_2$ then one of them must be equal to zero, which has been disallowed by the assumptions. Thus one obtains,

$$B(\sigma + \lambda) = C \sigma. \quad (3.8)$$

A similar exercise with (3.5) leads to the equation,

$$C(1 + \lambda) = B r. \quad (3.9)$$

Hence, by combining these as simultaneous equations one can deduce there is a quadratic,

$$\lambda^2 + (1 + \sigma)\lambda - (r - 1)\sigma, \quad (3.10)$$

which has the solutions (given by the quadratic formula),

$$\lambda_{1,2} = -\frac{1+\sigma}{2} \pm \frac{1}{2} \sqrt{(1+\sigma)^2 + 4(r-1)\sigma}.$$

It can be seen that the solutions' stability is fundamentally influenced by the value of r . If $r < 1$ then both λ_1 and λ_2 are negative and hence X and Y directions are both stable (along with the Z-direction) whereas if $r > 1$ then one of the λ_s will be positive making one of the directions in the X or Y direction unstable. This can be shown intuitively by a 'cartoon graph':

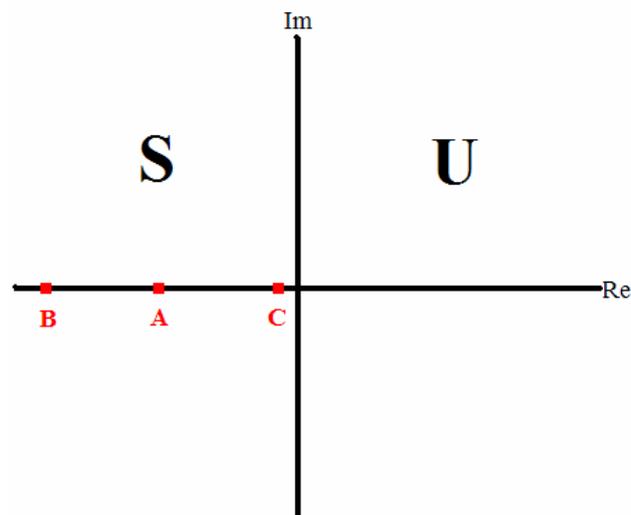


Figure 3.1: Diagram illustrating the stability of the steady state solution.

In the graph A represents the point $-(1+\sigma)$, and B and C represent λ_1 and λ_2 respectively. (λ_1 is the root where we take the negative square root and λ_2 is the root where we take the positive square root).

One should be able to see, therefore, that as r increases C gets closer and closer to the origin, and when $r = 1$ C sits on the origin, so becomes a 'saddle' root (neither stable nor unstable), and then when $r > 1$, C becomes the unstable root.

The deduction, therefore, is that for $r < 1$ the point C_0 is a stable manifold. Furthermore, we have shown that no matter the value of r the Z-axis is always stable for the point C_0 (any value starting on the Z-axis move towards the point C_0).

(3.2) The Stability of the point C_1 and C_2

Now to the other steady state solution, C_1 (then use the symmetry of the system to get the properties of C_2). Using (3.1), (3.2) and (3.3) we deduce the linearized equations below:

$$\frac{dx}{dt} = \sigma(\varepsilon(y) - \varepsilon(x)), \quad (3.11)$$

$$\frac{dy}{dt} = [\varepsilon(x) - \varepsilon(y)] - \varepsilon(z)\sqrt{b(r-1)}, \quad (3.12)$$

$$\frac{dz}{dt} = [\varepsilon(y) + \varepsilon(x)]\sqrt{b(r-1)} - \varepsilon(z)b. \quad (3.13)$$

As equation (3.11) is equivalent to (3.4), it would seem logical to assume that the solutions are of the same form as well. One could deduce this assumption by following the same arguments as before, but to save time the author will immediately assume it, and see what solutions emerge. So, let $\varepsilon(x) = \alpha e^{\mu t}$, $\varepsilon(y) = \beta e^{\mu t}$ and $\varepsilon(z) = \gamma e^{\mu t}$, where $\mu \neq 0$ and $t \geq 0$. Substitution into (3.11), (3.12) and (3.13) leads to these three equations:

$$\alpha(\sigma + \mu) = \sigma\beta, \quad (3.14)$$

$$\beta(1 + \mu) = \alpha - \gamma\sqrt{b(r-1)}, \quad (3.15)$$

$$\gamma(b + \mu) = (\alpha + \beta)\sqrt{b(r-1)}. \quad (3.16)$$

One can, as before, do back substitution to remove all the constants and define a cubic equation for μ . The author will show the critical step to help the reader follow the computation; $[\sigma - (\sigma + \mu)(1 + \mu)](b + \mu) = (2\sigma + \mu)b(r-1)$ is what should be obtained after cancelling all the constants, and hence leads to the cubic equation below

$$\mu^3 + (1 + b + \sigma)\mu^2 + b(r + \sigma)\mu + 2b(r-1)\sigma = 0. \quad (3.17)$$

This can be solved analytically, and has the solutions,

$$\mu = \sqrt[3]{q + \sqrt{q^2 + (\bar{r} - p^2)^3}} + \sqrt[3]{q - \sqrt{q^2 + (\bar{r} - p^2)^3}} + p,$$

where $p = -\frac{(1+b+\sigma)}{3}$, $q = -\frac{(1+b+\sigma)^3}{27} + \frac{[b(1+b+\sigma)(r+\sigma) - 6b(r-1)\sigma]}{6}$ and $\bar{r} = \frac{b(r+\sigma)}{3}$. This is particularly messy and complicated and not very insightful. So

one tries investigating varying r , to compare with previous result for C_0 , it can be noted that all three roots will have negative real parts if

$$r < \frac{\sigma(3+b+\sigma)}{\sigma-b-1}, \quad (3.18)$$

This is simply determined by considering (3.17) and let $\mu=0+i\omega$, so that two equations are obtained, namely;

$$\omega^3 - \omega b(r+\sigma) = 0 \text{ and } \omega^2(1+b+\sigma) - 2b\sigma(r-1) = 0,$$

and the first one is used to deduce that $\omega = 0, \pm\sqrt{b(r+\sigma)}$, and substitute this into the second equation to find that $r=1$ or $r = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$. Thinking about it, having $\mu=0+i\omega$ implies an oscillatory motion in the imaginary plane, if r is less than the values prescribed then we get exactly as was stated, negative real parts for μ .

Now, define $r_H = \frac{\sigma(3+b+\sigma)}{\sigma-b-1}$, as Sparrow did, then if one restricts oneself to just varying r with $\sigma=10$ and $b=\frac{8}{3}$, as Lorenz first did in his paper it can be deduced that $r_H = \frac{470}{19} \approx 24.74$, which means that for $1 < r < r_H$ both C_1 and C_2 are stable, and if $r > r_H$ then the two complex solutions will have positive real parts, and so the equilibria become unstable. At $r = r_H$ there is a sub critical Hopf Bifurcation, (see Chapter 2). This has not been discovered analytically but has been supported by the numerical calculations done and the reader will see more on this in Chapter 4.

(3.3) Summary

So, in this chapter it has been deduced that the steady state solutions have the following characteristics:

1. $0 < r < 1$, then C_0 is stable and C_1, C_2 do not exist.
2. $1 < r < \frac{\sigma(3+b+\sigma)}{\sigma-b-1} = r_H$, then C_0 becomes unstable in one direction, and C_1, C_2 are stable.
3. $r = r_H$, there is a sub critical Hopf bifurcation.
4. $r > r_H$, then C_0, C_1 and C_2 all become unstable.

All the above can be illustrated nicely on another ‘cartoon graph’ below:

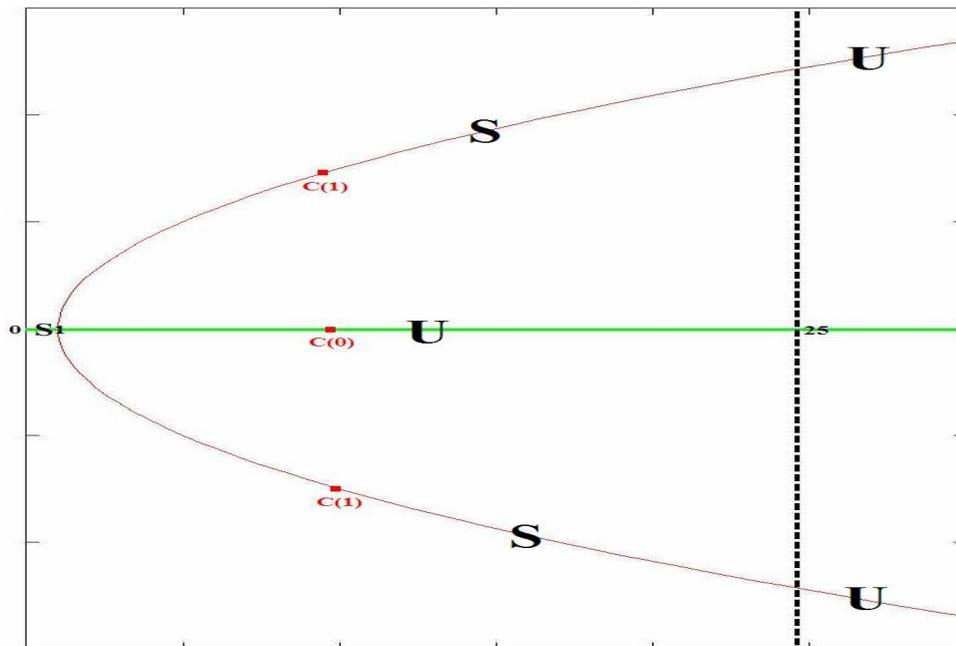


Figure 3.2: Diagram showing the points discovered in this chapter.

Hopefully, it can be seen by this simple diagram that if one takes the abscissa to be r and the ordinate to be x then all the steady solution points are there, and the author has represented the change of stability by the dashed line. The diagram, also, has the stability labelled (S for stable and U for unstable).

Please note that in (3.2) the author only studied one of the points when considering the bifurcation (as stated at start of the section), this was due to the symmetry of the Lorenz equations (see chapter 1). Hence one does not need to study both as just the one has given the information on both solutions.

4 The Numerical Approach

(4.0) Introduction

The aim of this chapter is to enlighten the reader to the vast magical world of the Lorenz Equations, as this cannot be done analytically the author must take the numerical approach. The reader should, by the end of this chapter, see how the author reached some of the earlier properties, but also see what else happens when one varies the parameters of the equations, and maybe ignite some more complex questions about the system.

The reader should note that all the MatLab codes used to do the numerical calculations in this paper are in appendix A. If more information is needed on MatLab the author directs the reader to, *A guide to MatLab, Highman 2002*.

The author wants to mention that in all the graphs of the Lorenz system seen in this chapter, and indeed this paper, appear to have *crossing* trajectories, these are not in reality crossing but are just due to the projection of a 3D image onto a 2D plane. If one had a 3D model of the trajectories one would see that no trajectories cross.

(4.1) Approaching Chaos

The author will first concentrate on varying one variable, r , as Lorenz did. The reason for this approach is to see what happens as one closes into the r_H as defined in the last chapter, this will give insight into the behaviour of the whole system at differing variables.

Let $\sigma = 10$ and $b = \frac{8}{3}$, hence, as defined by chapter 3, $r_H = \frac{470}{19} \approx 24.74$. By setting $r = 22.2$, the theory states that there should be steady state solutions at

$$\left(\pm \sqrt{\frac{848}{15}}, \pm \sqrt{\frac{848}{15}}, \frac{106}{5} \right) \approx (\pm 7.52, \pm 7.52, 21.2). \quad (4.1)$$

With r set, the theory implies that the system should be stable (that is any trajectory should in fact tend to one of the given steady state solutions). To see if the numerical solutions tie in with the theory the author used MatLab to draw the graphs of the trajectories for the above variables. The author has set the initial conditions to (10, 0, 10), but this should not effect the steady state solutions, just the amount of time needed to reach it.

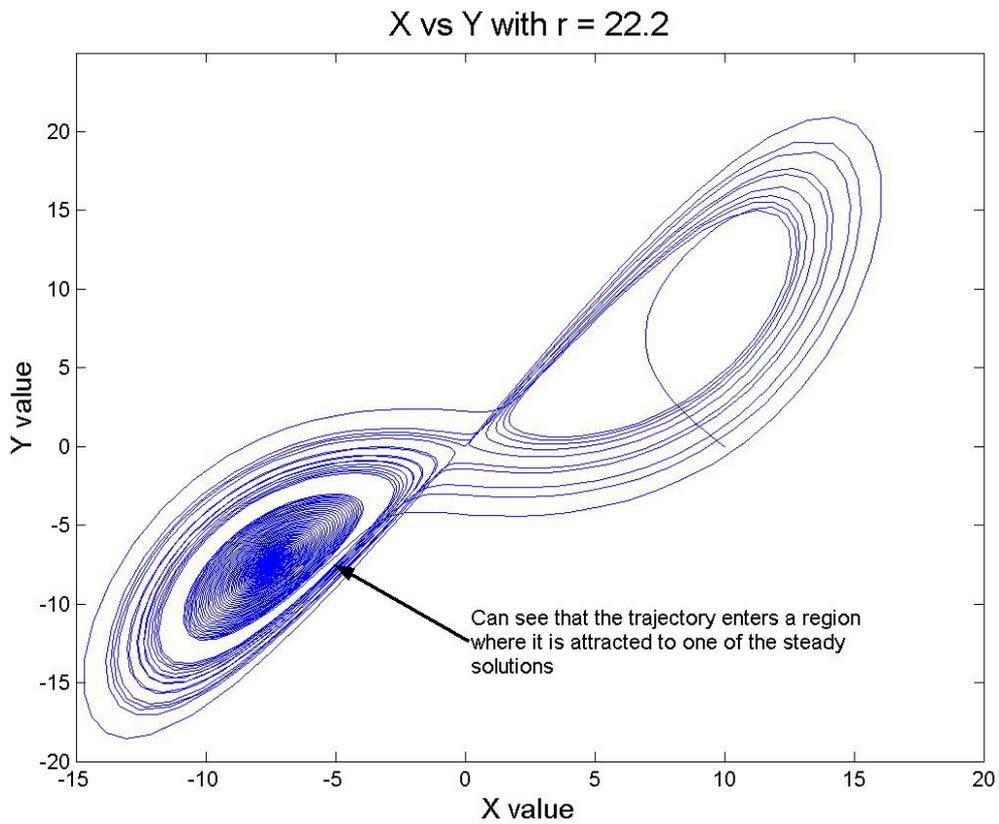


Figure 4.1: Graph of X vs. Y to see the steady state solution.

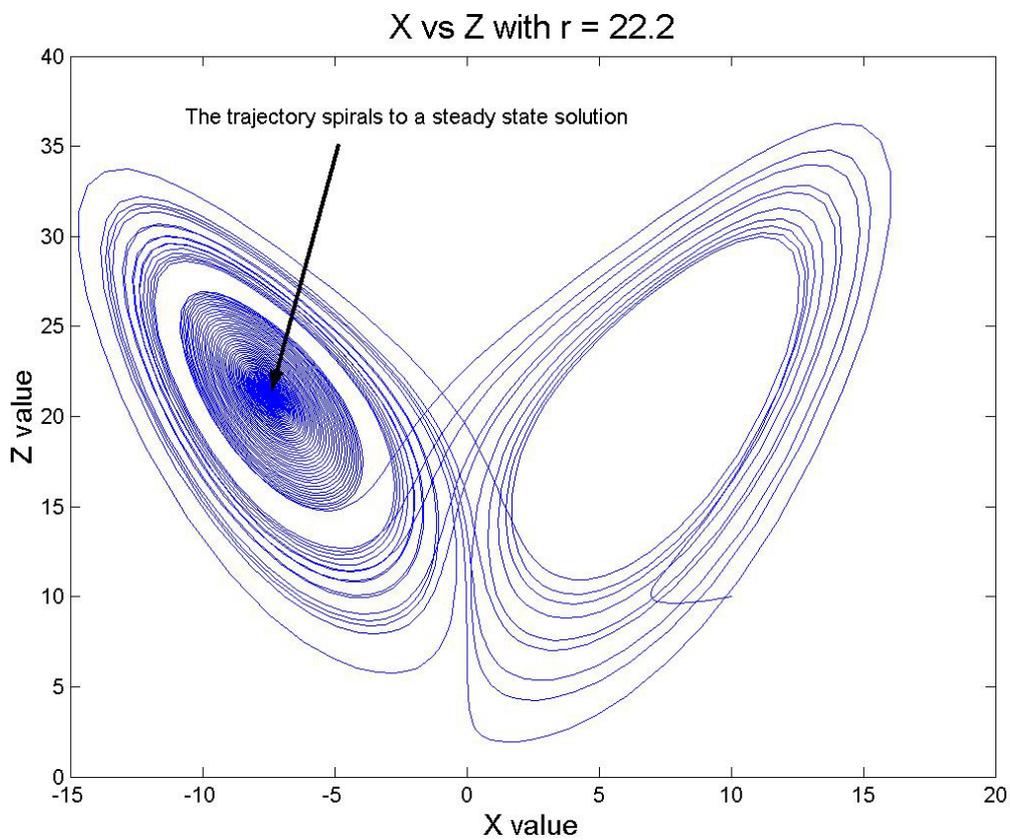


Figure 4.2: Graph of X vs. Z to see the correlation of theory and numerical analyses.

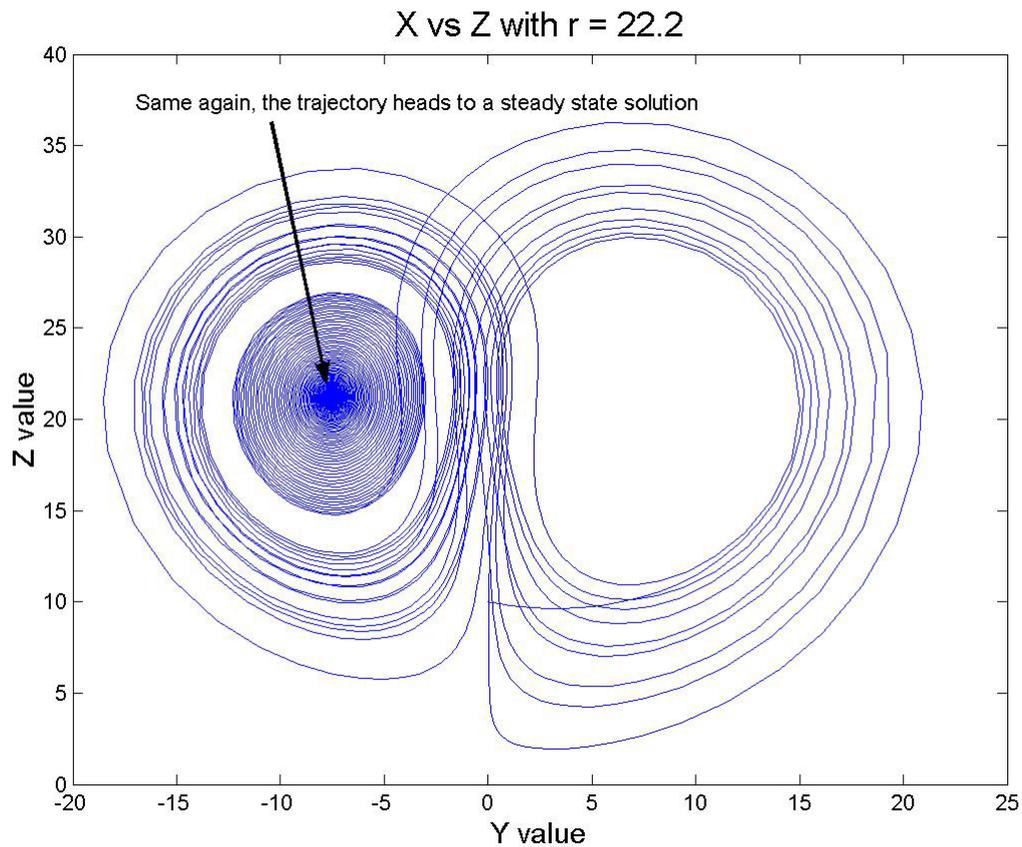


Figure 4.3: Graph of Y vs. Z completing the vision of Lorenz system.

It may seem excessive to include all the different plane views for this value of r , (and indeed it shall not be done again, the author from here on will concentrate to the X - Z plane view although there are two other planes to consider), but the author wanted the reader to see the different angles of the solutions to get a real *feel* for system and to emphasise how neatly the numerical and analytic solutions tie together. By looking at the graph the author hopes one can see that the steady state solutions are exactly those predicted by the theory.

One thing the theory did not suggest was the existence of a *pre-chaotic state*. Figure 4.4 is a perfect illustration, with r set to 22.2, the author examined time versus X to see what it shows, and one can see the *pre-chaotic state* where the solution seems to be chaotic but then settles down to oscillatory and finally tends to the steady state solution.

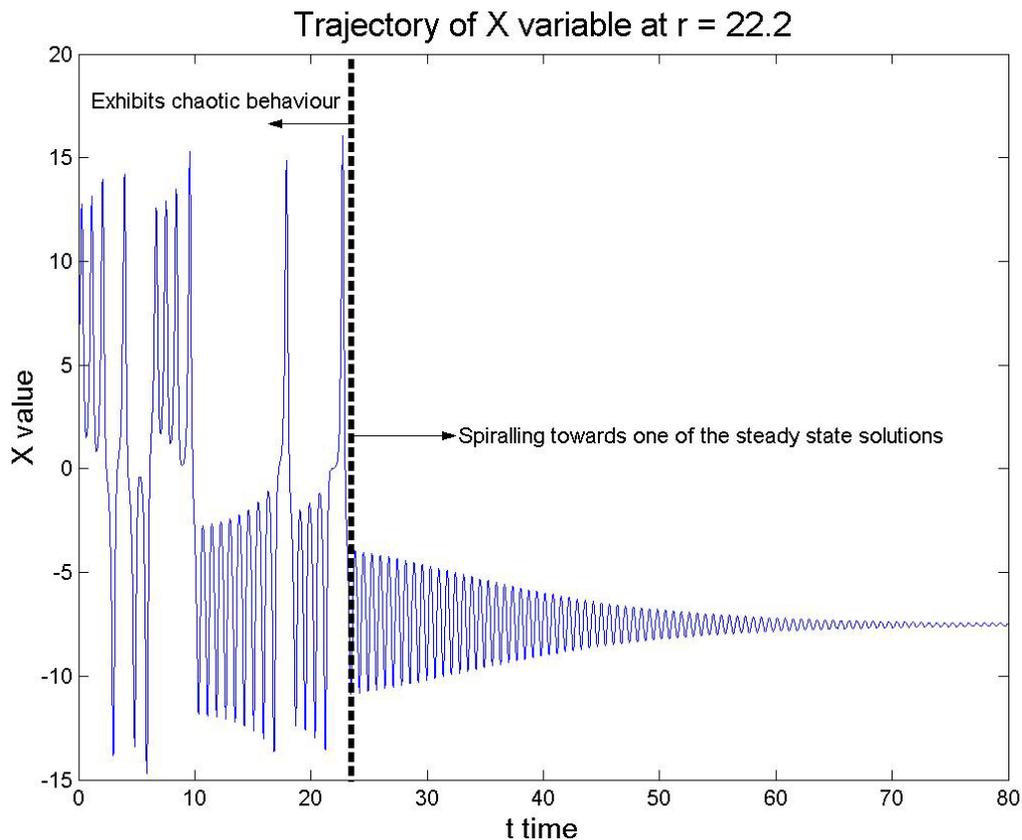


Figure 4.4: Graph showing pre-chaos characteristics.

(4.2) Shifting the chaotic boundary

In the previous section the author chose $\sigma = 10, b = \frac{8}{3}, r = \frac{111}{5}$ to show some of the characteristics of the system, one now knows that $r_H \approx 24.74$ for those variables, hence for $r < r_H$ the system has two stable steady solutions and for $r > r_H$ then the system has unstable steady solutions. One can vary r_H by simply varying σ and b , using formula (3.18), this will change chaos threshold.

If one starts with $\sigma = 10, b = \frac{8}{3}$ and $r = 28$, say, then it is clear that the system is in chaos, now if the system then, for some reason, changes and the value of r decreases, does the chaos disappear, and if so is it exactly at r_H or at some value above or below it?

Well the answer is that chaos has to disappear at some point, otherwise chaos would then reign over all r and it is known that this is not the case, the question on when it disappears is a lot more difficult. In the last chapter the reader may remember that the author stated that at r_H there is a subcritical Hopf bifurcation, this is what affects the chaos boundary. The subcritical Hopf *draws* the chaos back into the *stable* region, but in this region the trajectories are unstable and so could return to the steady state solution at any time. The author would like to show a simple graph / diagram for this

however it is not a simple situation so one cannot be put in, however the author refers the reader to Sparrow 1982 which gives good diagrams and explanations of this phenomenon.

(4.3) Sensitivity to Initial conditions

One property of the Lorenz equations not yet discussed in this paper is the sensitivity to initial conditions. It is the most fundamental property of the system and the reason the author has waited until now to discuss it is because one needs to understand the complexity of the system before one can appreciate the importance of this property.

If one were to start any system from two extremely close points then common sense would tell us that the trajectories should remain extremely close for all time. However, this is not the case with the Lorenz equations, in fact the contrary is true, no matter how close you put two starting points, within a relative time scale the trajectories will part and take completely independent paths.

If one starts the Lorenz equations from $X_1 = X_2 = 10$, $Y_1 = Y_2 = 0$, $Z_1 = 10$ and $Z_2 = 10.000000000001$ then for the first 25 time units the trajectories appear to be identical but beyond 30 time units they are totally unrelated to each other.

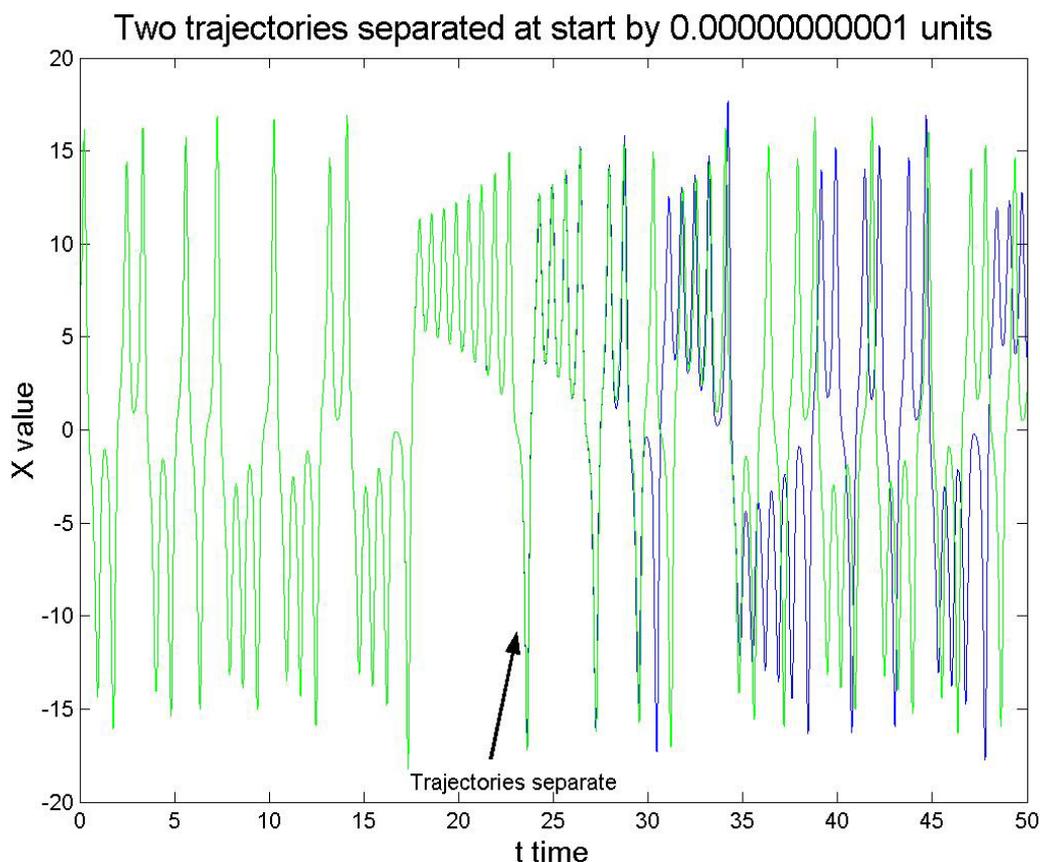


Figure 4.5: Graph showing the separation of two trajectories separated at start by 10^{-10} units.

It is precisely this property that makes long term predictions of physical systems impossible. It devastates the classicalists' view of the world. The rate at which the trajectories separate is phenomenal and is known as the Liapunov exponent, in this case it is $e^{\lambda t}$, (this only applies at the initial separation).

(4.4) Chaos reigns

It is known that chaos begins once the system passes the critical point r_H , but does it continue forever? Well yes in a sense it does, once the system is beyond r_H chaos remains forever. There are a few properties to mention which are period doubling, homoclinic explosions and intermittent chaos. All these properties will not be discussed formally in this paper as is far too in depth, however in the next chapter the author will explain what they are and why they need more exploration.

The classic view of this system is below, and the parameters are $\sigma = 10, b = \frac{8}{3}, r = 28$, and starting value of (0.1, 0.1, 0.1). The view seen is the same for all the chaos exhibiting parameters of the Lorenz equations. The author did not discuss small values of r because the system is very easily resolved at such values and wanted to spend more time on the more interesting properties of the system.

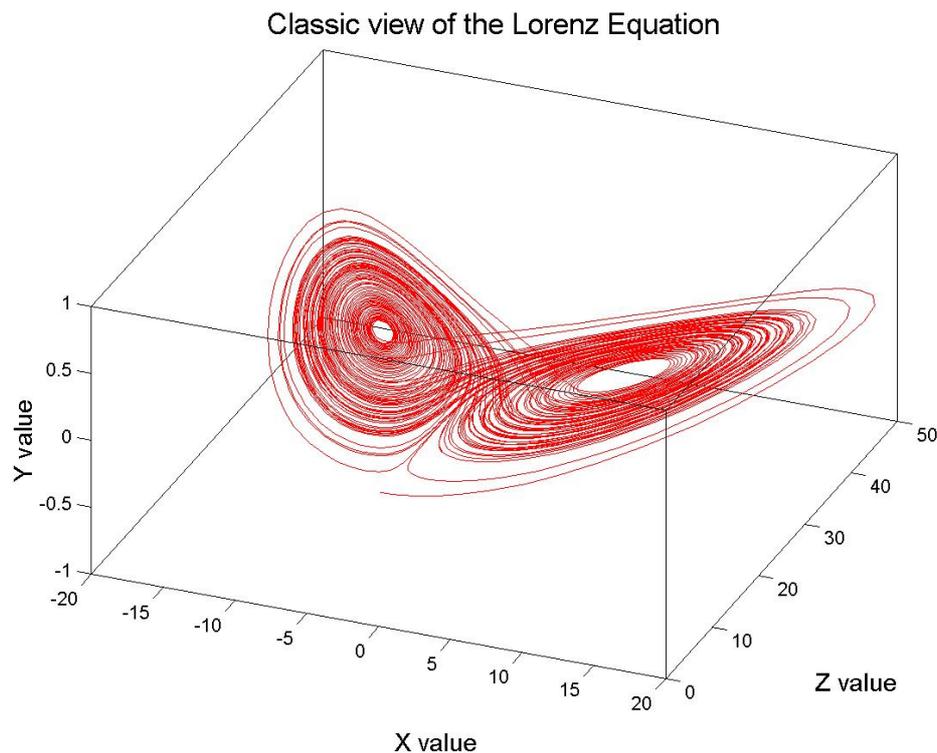


Figure 4.6: The classic 'Lorenz Butterfly' showing the generic shape of all solutions in the chaos region of the system.

5 Further Interesting Properties

(5.0) Preliminaries

This chapter is to highlight some properties the author has come across in the study of the Lorenz equations, but has not had time to fully develop the analysis or proofs. The properties are interesting so need to be mentioned but to get full descriptions the reader must refer to Sparrow 1982, Gleick 1987 or Hale and Kocak 1991, indeed if this chapter inspires the reader to investigate further, those books are a good place to start.

The reader must note that in this chapter the author will make statements without proof or indeed the total knowledge to prove the statement, the author just knows the properties exist by exploration of the system.

(5.1) Period Doubling Windows

The author came across this property when investigating various values of r well above that of r_H . The property is essentially where the trajectories enter a stable orbit around the two steady state solutions, even though the steady state solutions are unstable. The author has discovered three such windows, but there maybe more. The windows the author has discovered may not be one hundred per cent accurate but will illustrate the property very well.

The first such window the author discovered is for values $99.53 < r < 100.79$. The window has been defined with $\sigma = 10$ and $b = \frac{8}{3}$, the window size is calculated from numerical computations in MatLab. In this period doubling window the trajectories move around one steady state solution and then twice around the second and back to the first once more (denoted [1-2-2]), as seen in figure 5.1.

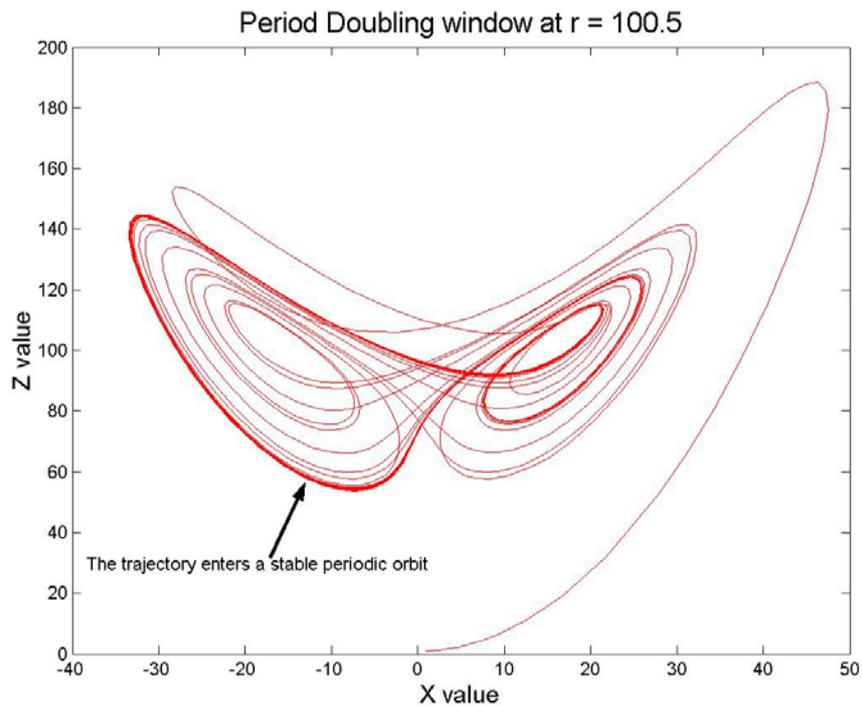


Figure 5.1: Graph illustrating the stable orbit trajectory when $r = 100.5$.

As the value of r decreases the stable orbit widens or *doubles*, creating the orbit to do as above but then repeat it once more before returning to the original orbit (denoted $[1-2-2-1-2-2]$ or $[1-2-2]^2$). The system continues to *cascade* in this way to the lower bound of the window.

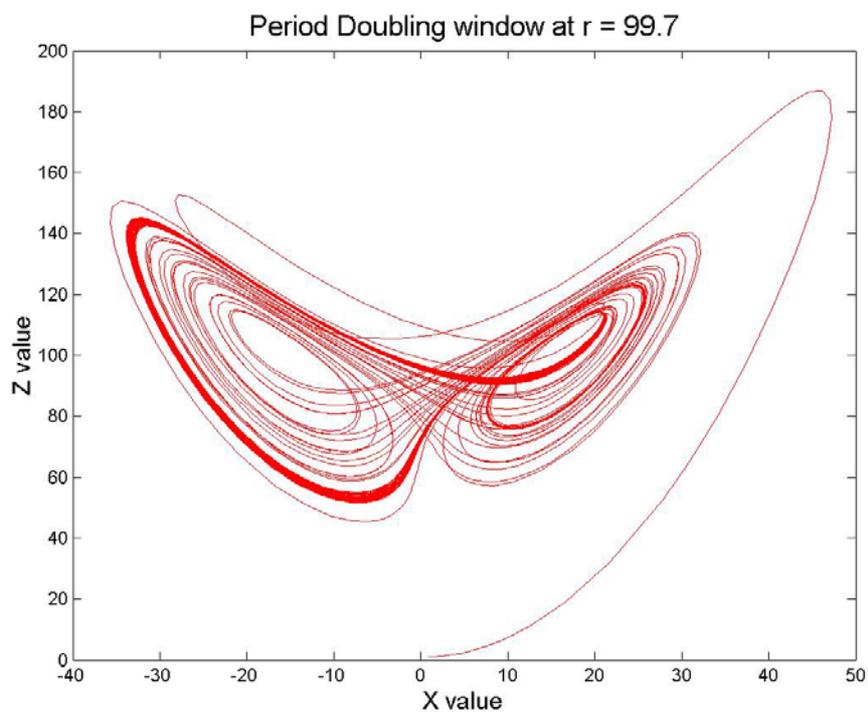


Figure 5.2: A graph illustrating the doubling effect of the stable orbit as r is decreased to $r = 99.7$.

If one continues to decrease the value of r still further not only does the *cascade* effect still occur but one also notices that the orbits reflect and therefore the trajectories orbit the other steady state solution twice whereas before it would have been just one (i.e. [1-2-2] to [2-1-2]), as demonstrated in figure 5.3.

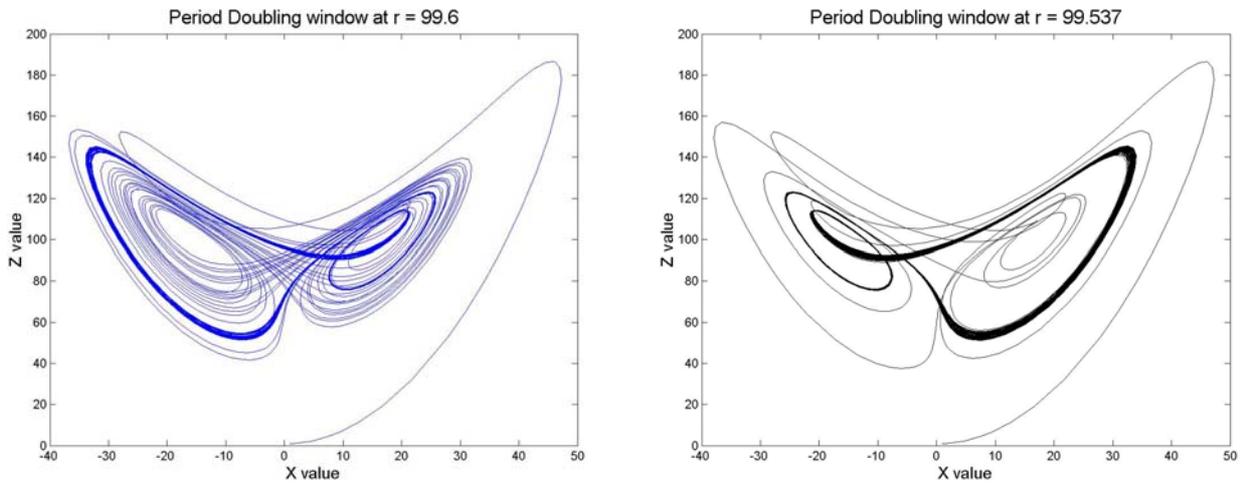


Figure 5.3: Graphs showing that a small change in r will result in the orbits flipping.

The second period doubling window the author noticed is $145 < r < 166$. In this region much more complicated scenarios occur, however there is not enough time to discuss them all in this paper. The author will highlight the fact that this period doubling window is symmetric about the two steady state solutions (denoted [2-2-2]), as figure 5.4 illustrates.

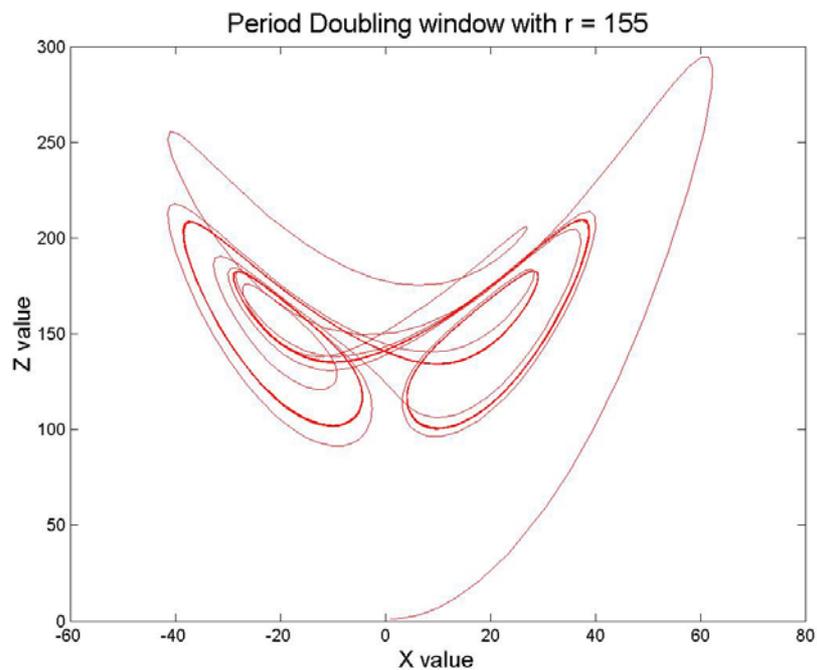


Figure 5.4: Graph of second period doubling window at $r = 155$.

The third and final period doubling window is $214.36 < r$. There is no upper bound as the author tried many values and always found a period doubling. In fact, as one increases r further and further the orbits become more and more stable periodic, as figure 5.5 should show.

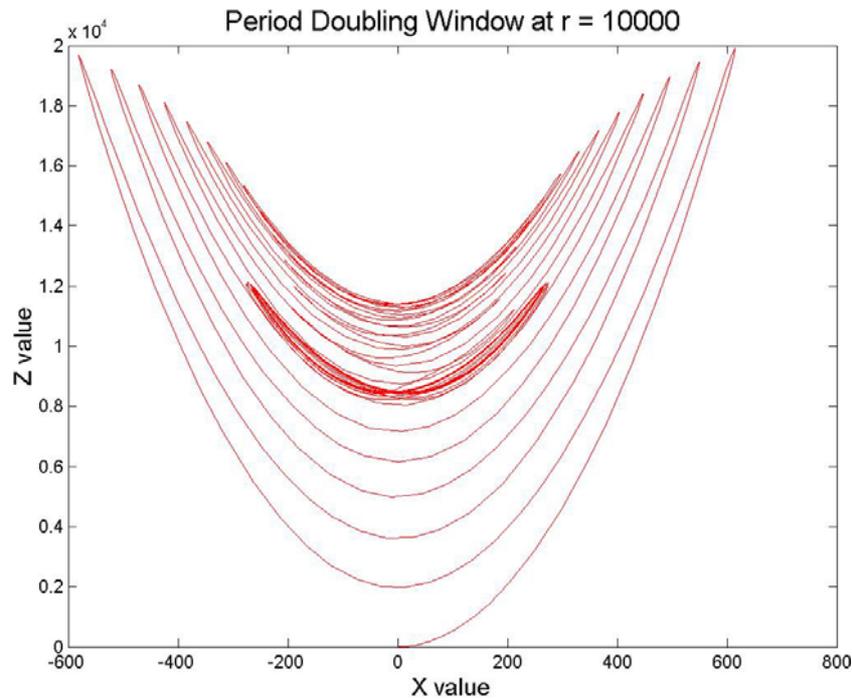


Figure 5.5: A graph of the last period doubling window at a very large value of r .

(5.2) Intermittent Chaos

In the search for answers to the last section the author also noticed a *quirky* property. It seems that when the trajectories are close to a period doubling window (below the window or above) the trajectories exhibit intermittent chaos. This is where the trajectory can be stable oscillatory and then suddenly switches to chaos and then revert back to the oscillatory trajectory. Moreover, as one moves further and further from the windows the intermittent chaos seems to become more and more frequent until it becomes dominant and then pure chaos returns. The author does not have any proof of why this happens but it is an interesting and unusual property of the Lorenz equations.

In the eyes of the author what happens is: Chaos exists and as one moves towards the periodic window intermittent chaos appears until chaos ceases leaving periodic orbits in the window and then as one moves out of the window the intermittent chaos returns once more and finally as one increases still further the chaos totally takes over once again. Hopefully, figure 5.6 should show the reader what intermittent chaos can look like if they wish to investigate the property further.

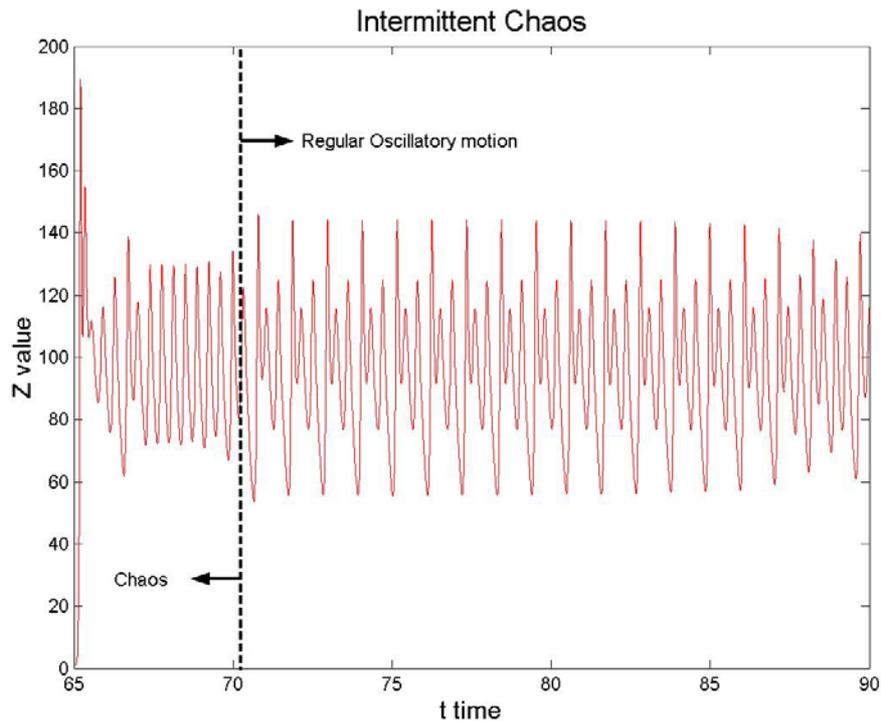


Figure 5.6: A graph showing intermittent chaos.

(5.3) Homoclinic Explosions and Small values of b

The author does not have any real knowledge of homoclinic explosions, however whilst researching this project there was an awful lot of information on the subject and thus would suggest the reader to study them if the reader wants to study the Lorenz equations still further. The author can suggest Sparrow 1982 and Gleick 1987 as two good books to start on.

The author noticed a very disturbing graph when studying the Lorenz equations using MatLab, and investigating different values of σ and b . The graph that was derived did not look like any of the others and started the search an explanation. The author searched all the literature and found very little to start with but then when looking through the Sparrow book 1982 the author found the answers. If the parameter (5.1) is greater than $\frac{2}{3}$ then there is a stable symmetric orbit that exists $\forall r$ and which winds around the z-axis.

$$\pi = \frac{\sigma + 1}{b + 2}. \quad (5.1)$$

This property is strange, and according to Sparrow 1982, there are a few other restrictions to the strange behaviour. If $\pi > 1$ then we get as described above and if $\frac{2}{3} < \pi < 1$ then there is a pair of non-symmetric, non-stable periodic orbits which do not wind around the z-axis. When this is the case the Hopf Bifurcation does not occur

and C_1 and C_2 remain stable $\forall r$. The author wants to mention that the above parameter (5.1) and the above statements have been taken from Sparrow 1982: 149.

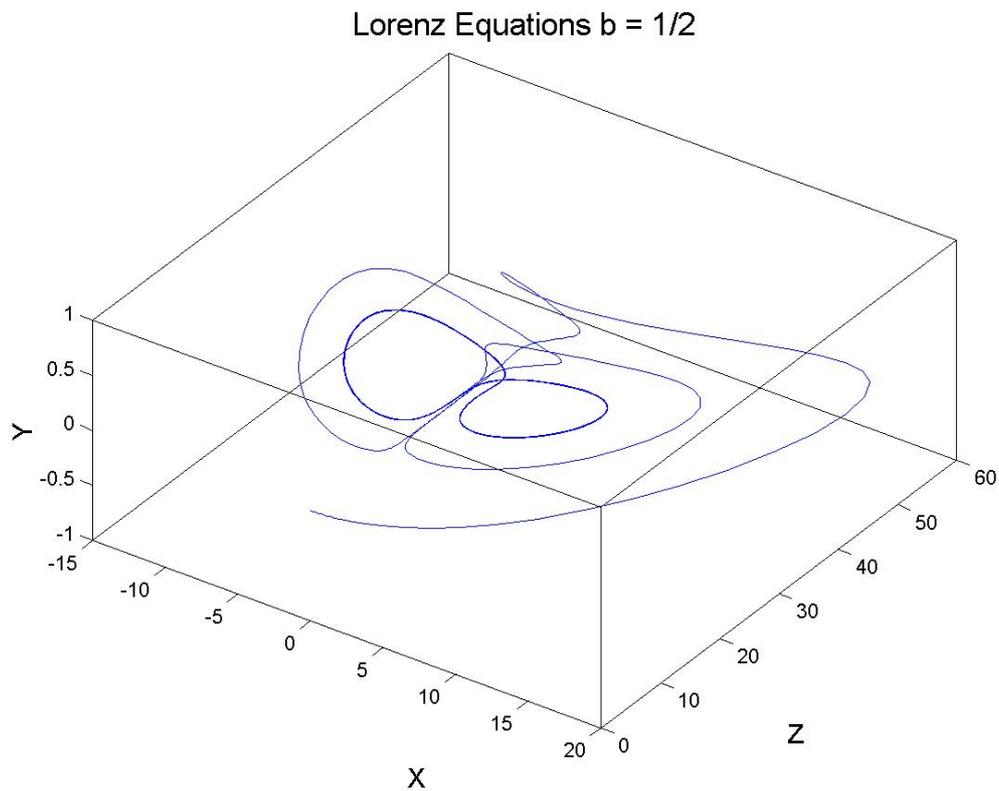


Figure 5.7: The graph that revealed more secrets of the Lorenz equations, the parameters are set to $\sigma = 10, r = 28$ and $b = \frac{1}{2}$.

Figure 5.7 is the graph that aroused the author's interest in small values of b , and as such lead to the discovery of the parameter (5.1). One can clearly see the symmetric, stable orbit around the z -axis.

6 Conclusion

(6.0) Conclusion

The author hopes that from this short paper it is clear that the system defined by Lorenz in the 1960s is a most detailed and intricate system. It has many features beyond the scope of this paper as mentioned many times before and if the reader still has a thirst for more then the author would advise reading all of the references as they will widen the mind to this system and its properties.

The Lorenz system has been studied in abundance now and it is assumed that most of the mysteries of the system have been explained. One would use this system as a simple introduction to chaos theory before moving into much more complex systems. Indeed there have been many more, higher dimensional, extensions to Lorenz's system which have been studied and have similar properties to this dimensional version although details are far more complicated.

In writing this paper the author has come to accept that chaos is the *normal state* for physical systems, and that much more work is needed to understand what is going on and how to compute these systems. For now it seems we will be restricted to numerical analysis of the systems, which can still explain a great deal about the system.

Hopefully, the reader will have heard of the *butterfly effect* before reading this paper, this refers to the sensitivity to initial conditions. It is precisely this system that has led to this popular phrase, although it is usually explained using an old folk poem in which a misplaced nail causes a kingdom to fall (Gleick 1987: 23).

(6.1) Afterword

The author would like to study the system further to complete the picture, but it would require much more time and more skills not yet attained by the author. With this paper have come more questions than answers; as one paper in 1963 completely changed the world's view on deterministic equations is there another shrewd mind thinking up some other paper where our view of the world will change again. The author suspects that the story is not yet fully told, and as a subject it will expand and diversify for many years to come and one day the whole truth of chaos may be revealed for all to see.

7 References

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All figures used in the text were produced by the author using MatLab 6.1.

8 Appendix A

Below is the MatLab code used throughout this project and has been modified several times to produce different graphs and different computations.

```
% Lorenz ODE solving function

tspan = [0 100]; % Solve for 0<=t<=100
options = odeset('MaxStep',0.1);
y0 = [1;0;1]; % Initial conditions
[t,y] = ode45(@LORENZDE1,tspan,y0,options);

plot(y(:,1),y(:,3),'red') % Plots (y_1,y_3) phase space

xlabel('X value','fontsize',14)
ylabel('Y value','fontsize',14)
zlabel('Z value','fontsize',14)
title('Lorenz Equations','fontsize',16)
```

The code below is the function file LORENZDE1 which was used to change the parameters in the numerical analysis.

```
function dydt = lorenzde(t,y)

%LORENZDE LORENZ EQUATIONS

% dy/dt = lorenzde(t,y)
% dy(1)/dt = a*(y(2)-y(1));
% dy(2)/dt = r*y(1)-y(2)-y(1)*y(3);
% dy(3)/dt = y(1)*y(2)-b*y(3);

dydt = [10*(y(2)-y(1)); 28*y(1)-y(2)-y(1)*y(3); y(1)*y(2)-(8/3)*y(3)];
```

The third code is what can be used to see the sensitivity of initial conditions. It solves the Lorenz equations twice with different initial values and then plots them together so that it can be observed.

```
% Lorenz ODE solving function
tspan = [0 50]; % Solve for 0<=t<=100
options = odeset('MaxStep',0.1);
y0 = [10;0;10]; % Initial conditions
[t,y] = ode45(@LORENZDE1,tspan,y0,options);
plot(t,y(:,1),'blue') % Plots (y_1,y_3) phase space
xlabel('t','fontsize',14)
ylabel('X value','fontsize',14)
zlabel('Y','fontsize',14)
title('Lorenz Equations','fontsize',16)
hold on
y1 = [10;0;10.00000000000001];
```

```
[t,y] = ode45(@LORENZDE1,tspan,y1,options);  
plot(t,y(:,1),'green')  
xlabel('t time','fontsize',14)  
ylabel('X value','fontsize',14)  
zlabel('Y','fontsize',14)  
title('Two trajectories separated at start by 0.00000000001 units','fontsize',16)  
hold off
```

These are the only codes needed to produce all the figures in the text and more besides. If the reader needs any more information the author suggests they read Highman, 2005.

Please note that in the numerical analysis used for this paper the Runge Kutta method was used, which employs 4th order accuracy, rather than an Euler simple 1st order accuracy. It is important to note this because it means that the numerics can be regarded as reliably accurate. Again if the reader requires more information the author suggest reading *Numerical Recipes in C: The Art of Scientific Computing*, 16, 710-714, it will be very helpful.